Functiones et Approximatio
*** (202*), 1–9*
doi: 10.7169/facm/1866

ON THE NUMBER OF DIVISORS OF THE LEAST COMMON MULTIPLES OF SHIFTED PRIME POWERS

Florian Luca, Francesco Pappalardi

Abstract: In this paper, we give the order of magnitude for the summatory function of the number of divisors of the least common multiple of $p^i - 1$ for $i = 1, 2, \ldots, k$ when $p \leqslant x$ is prime.

Keywords: divisors, primes, applications of sieve methods.

1. Introduction

In this paper, p is a prime number and $\tau(n)$ is the number of divisors of n. The Titchmarsh divisor problem is the upper bound

$$\sum_{p \leqslant x} \tau(p-1) = O(x)$$

proved by Titchmarsh in 1930 (see [9]). Since then, many authors found asymptotic expressions for the left–hand side above (see [5], [1]). In [7], it was shown that

$$\sum_{p \leqslant x} \tau_k(p-1) \asymp_k x(\log x)^{k-2},$$

where $\tau_k(n) = \#\{(a_1, \ldots, a_k) \in \mathbb{N}^k : a_1 \cdots a_k = n\}$ is the coefficient of $1/n^s$ in the expansion of $\zeta(s)^k$.

We recall that the notation $A \asymp B$ means that both $A = O(B)$ and $B = O(A)$ hold. The subscript (if present) indicates that the implied constants in the above O depend on the parameter from the subscript. For $k = 1$, $\tau_1(n) = 1$ and for $k = 2$, $\tau_2(n) = \tau(n)$.

In this article we look at the least common multiple of $p^i - 1$ for $i = 1, \ldots, k$ and of $p^i + 1$ for $i = 1, \ldots, k$ and we give upper and lower bounds for the average number of divisors of these expressions when p ranges over primes in $[1, x]$.

2020 Mathematics Subject Classification: primary: 11N56; secondary: 11N36

Theorem 1. *We have the following estimates:*

$$\sum_{p\leqslant x}\tau(\mathrm{lcm}[p^k-1,p^{k-1}-1,\dots,p-1])\asymp_k x(\log x)^{k-1},$$
$$\sum_{p\leqslant x}\tau(\mathrm{lcm}[p^k+1,p^{k-1}+1,\dots,p+1])\asymp_k x(\log x)^{k-1},$$

as $x\to\infty$.

Our results might have some practical applications to other areas. For example, the number of divisors of p^k+1 is related to the number of reduced Egyptian fractions of length 2 with denominator p^k (see [6] and [2]), whereas the number of divisors of $\mathrm{lcm}[p-1,\dots,p^k-1]$ is related to the exponent of the group $\mathrm{GL}_k(\mathbb{F}_p)$ of invertible $k\times k$ matrices with entries in the finite field with p elements.

2. The proof

We prove only the second estimate since the first one is similar (and slightly easier). Let

$$I_k:=\{2i+1:0\leqslant i\leqslant(k-1)/2\}\cup\{4i:1\leqslant i\leqslant k/2\}.$$

For example, for $k=5$, we have $I_5=\{1,3,4,5,8\}$, so $\#I=5$. In general, $\#I_k=(1+\lfloor(k-1)/2\rfloor)+\lfloor k/2\rfloor=k$. We omit the dependence on k in the set I_k and just write I.

We start with some considerations about the lcm. Note that if i is odd then $X^i+1=-((-X)^i-1)$, while if i is even then $X^i+1=(-X)^i+1=((-X)^{2i}-1)/((-X)^i-1)$. Hence,

$$X^i+1=\begin{cases}-\prod\limits_{d\mid i}\Phi_i(-X) & \text{if } i\equiv 1 \pmod 2,\\ \prod\limits_{\substack{d\mid 2i\\ d\nmid i}}\Phi_d(-X) & \text{if } i\equiv 0 \pmod 2.\end{cases}$$

Here, $\Phi_m(X)$ is the mth cyclotomic polynomial. It follows that if we write:

$$\mathrm{lcm}[p^k+1,p^{k-1}+1,\dots,p+1]=\pm\delta_k^{-1}\prod_{i\in I}\Phi_i(-p),$$

then $\delta_k\in\mathbb{N}$ is a divisor of

$$\prod_{\substack{i,j\in I\\ i\neq j}}\gcd(\Phi_i(-p),\Phi_j(-p)).$$

Since

$$\gcd(\Phi_i(-p), \prod_{j<i} \Phi_j(-p)) \mid i,$$

(see Lemma 6 in [8]), it follows that

$$\delta_k \mid (k!)^2.$$

Therefore, if we set:

$$f_k(X) = \prod_{i \in I} \Phi_i(-X),$$

we have that

$$\tau(\mathrm{lcm}[p^k+1, p^{k-1}+1, \dots, p+1]) \asymp_k \tau(f_k(p)).$$

The lower bound

Let $A := A_k$ be some number depending on k to be determined later. We have

$$\begin{aligned}
&\sum_{p \leqslant x} \tau(\mathrm{lcm}[p^k+1, p^{k-1}+1, \dots, p+1]) \\
&\qquad \geqslant \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2=1 \\ \tau(m) \leqslant (\log x)^A}} \sum_{\substack{p \leqslant x \\ \mathrm{lcm}[1+p^k, 1+p^{k-1}, \dots, 1+p] \equiv 0 \pmod m}} 1 \\
&\qquad = \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2=1 \\ \tau(m) \leqslant (\log x)^A}} \sum_{i=1}^{k(m)} \pi(x, m, a_i^*) \\
&\qquad = \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2=1 \\ \tau(m) \leqslant (\log x)^A}} \frac{k(m)\pi(x)}{\phi(m)} + E,
\end{aligned}$$

where

$$E = \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2=1 \\ \tau(m) \leqslant (\log x)^A}} \sum_{i=1}^{k(m)} \left| \pi(x, m, a_i^*) - \frac{\pi(x)}{\phi(m)} \right|.$$

In the above, $a_1^*, \dots, a_{k(m)}^*$ are all the residue classes modulo m representing solutions p to the congruence $f_k(p) \equiv 0 \pmod m$. To understand $k(m)$, we note first that it is a multiplicative function. So, let $q > k$ be a prime. We need to understand k_q, which is the number of solutions modulo q of $f_k(X) \equiv 0 \pmod q$. First, let us note that the solutions coming from $\Phi_i(-X) \equiv 0 \pmod q$ and

$\Phi_j(-X) \equiv 0 \pmod q$ are distinct for $i \neq j$ in I. Indeed, if they are not, then $\Phi_i(-X)$ and $\Phi_j(-X)$ have a common root modulo q. This leads to a double root of $(-X)^{ij} - 1 \pmod q$. Any double root of $(-X)^{ij} - 1 \pmod q$ is a root of its derivative $-ij(-X)^{ij-1} \pmod q$ and this is not zero since $q > k \geqslant \max\{i, j\}$. Furthermore, 0 is not a root of $\Phi_i(-X) \pmod q$ or $\Phi_j(-X) \pmod q$. Thus, indeed $\Phi_i(-X)$ and $\Phi_j(-X)$ have no common roots modulo q. Now since the group of invertible elements modulo q has a primitive root ρ_q, it follows that $\Phi_i(-X)$ has roots modulo q if and only if $2i \mid q-1$, and in this case it has exactly $\phi(2i)$ such roots. They are exactly the residues $-\rho^{((q-1)/2i)\lambda} \pmod q$, where $\lambda \in [1, 2i]$ is coprime to $2i$. Thus,

$$k(q) = \sum_{\substack{2i \mid q-1 \\ i \in I}} \phi(2i).$$

Clearly, $k(q) \leqslant (2k)^2$. This is for $q > k$. This shows that

$$k(m) \leqslant (2k)^{2\omega(m)} \ll_k \tau(m)^{c_k},$$

where we can take $c_k := 2\log(2k)/\log 2$. Since $\tau(m) \leqslant (\log x)^A$, we get that $k(m) \ll_k (\log x)^{Ac_k}$. An application of the Bombieri-Vinogradov Theorem

$$\sum_{Q \leqslant x^{1/3}} \max_{\substack{a \bmod Q \\ y \leqslant x}} \left| \pi(y, m, a) - \frac{\pi(y)}{\phi(m)} \right| \ll_B \frac{x}{(\log x)^B},$$

shows that

$$E = \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2 = 1 \\ \tau(m) \leqslant (\log x)^A}} \sum_{i=1}^{k(m)} \left| \pi(x, m, a_i^*) - \frac{\pi(x)}{\phi(m)} \right| \ll_B \frac{x}{(\log x)^{B - Ac_k}} \ll_{k,A} \frac{x}{(\log x)^2},$$

provided we choose $B := Ac_k + 2$. It remains to deal with the main term. This is

$$\pi(x) \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2 = 1 \\ \tau(m) < (\log x)^A}} \frac{k(m)}{\phi(m)} = \pi(x) \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2 = 1}} \frac{k(m)}{\phi(m)} - \pi(x) \sum_{\substack{m \leqslant x^{1/3} \\ \mu(m)^2 = 1 \\ \tau(m) > (\log x)^A}} \frac{k(m)}{\phi(m)}. \qquad (1)$$

The first sum is, by a Tauberian theorem (see Theorem 4 in [4]),

$$\begin{aligned}
\pi(x)\sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1}}\frac{k(m)}{\phi(m)} &= C_k(1+o(1))\pi(x)\prod_{q\leqslant x^{1/3}}\left(1+\frac{k(q)}{q-1}\right)\\
&\asymp_k \pi(x)\exp\left(\sum_{q\leqslant x^{1/3}}\frac{k(q)}{q}+O\left(\sum_{q\geqslant 1}\frac{k(q)}{q^2}\right)\right)\\
&\asymp_k \pi(x)\exp\left(\sum_{k<q\leqslant x^{1/3}}\frac{\sum_{\substack{i\in I\\ 2i\mid q-1}}\phi(2i)}{q}+O_k(1)\right)\\
&= \pi(x)\exp\left(\sum_{i\in I}\phi(2i)\sum_{\substack{k<q\leqslant x^{1/3}\\ q\equiv 1 \pmod{2i}}}\frac{1}{q}+O_k(1)\right)\\
&= \pi(x)\exp\left(\sum_{i\in I}\phi(2i)\left(\frac{\log\log x^{1/3}}{\phi(2i)}+O_k(1)\right)\right)\\
&\asymp_k \pi(x)\exp\left(\#I\log\log x\right)\\
&\asymp_k x(\log x)^{\#I-1}\asymp_k x(\log x)^{k-1}. \qquad (2)
\end{aligned}$$

The second sum is

$$O_k\left(\pi(x)\sum_{\substack{m\leqslant x^{1/3}\\ \mu^2(m)=1\\ \tau(m)>(\log x)^A}}\frac{\tau(m)^{c_k}}{\phi(m)}\right).$$

To estimate this, we proceed as follows. First,

$$\begin{aligned}
\sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1}}\frac{\tau(m)^{c_k}}{\phi(m)} &\leqslant \prod_{q\leqslant x^{1/3}}\left(1+\frac{2^{c_k}}{q-1}\right)\\
&\ll_k \exp\left(\sum_{k<q\leqslant x^{1/3}}\frac{2^{c_k}}{q}+\left(\sum_{q\geqslant 1}\frac{2^{c_k}}{q^2}\right)\right)\\
&\ll_k \exp\left(2^{c_k}\log\log(x^{1/3})+O_k(1)\right)\\
&\ll_k (\log x)^{d_k},
\end{aligned}$$

where $d_k = 2^{c_k}$. Thus,

$$\sum_{\substack{m\leqslant x^{1/3}\\ \mu^2(m)=1\\ \tau(m)>(\log x)^A}} \frac{1}{\phi(m)} \ll (\log x)^{d_k-Ac_k}.$$

Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1\\ \tau(m)>(\log x)^A}} \frac{k(m)}{\phi(m)} &\ll_k \sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1\\ \tau(m)>(\log x)^A}} \frac{\tau(m)^{c_k}}{\phi(m)}\\
&\ll_k \left(\sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1}} \frac{\tau(m)^{2c_k}}{\phi(m)}\right)^{1/2} \left(\sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1\\ \tau(m)>(\log x)^A}} \frac{1}{\phi(m)}\right)^{1/2}\\
&\ll_k \left(\prod_{k<q\leqslant x^{1/3}} \left(1+\frac{2^{2c_k}}{q-1}\right)\right)^{1/2} (\log x)^{d_k/2-Ac_k/2}\\
&\ll_k \exp\left(\frac{1}{2}\sum_{k<q\leqslant x^{1/3}} \frac{2^{2c_k}}{q} + O_k(1)\right)(\log x)^{d_k/2-Ac_k/2}\\
&\ll_k \exp\left(2^{2c_k-1}\left(\log\log x + O_k(1)\right)\right)(\log x)^{d_k/2-Ac_k/2}\\
&\ll_k (\log x)^{d_k^2/2+d_k/2-Ac_k/2}.
\end{aligned}$$

Choosing $A > (d_k^2 + d_k)/c_k$, the last upper bound above becomes $O_k(1)$. Thus, with this choice, the second term on the right–hand side in (1) is $O_k(\pi(x))$. Thus,

$$\pi(x) \sum_{\substack{m\leqslant x^{1/3}\\ \mu(m)^2=1\\ \tau(m)<(\log x)^A}} \frac{k(m)}{\phi(m)} \gg x(\log x)^{k-1},$$

so that

$$\sum_{p\leqslant x} \tau(\mathrm{lcm}[p^k+1, p^{k-1}+1, \ldots, p+1]) \gg x(\log x)^{k-1}.$$

The upper bound

First of all

$$\begin{aligned}\sum_{p\leqslant x}\tau(\mathrm{lcm}[p^k+1,p^{k-1}+1,\ldots,p+1]) &\ll_k \sum_{p\leqslant x}\tau(f_k(p))\\ &\leqslant \sum_{n\leqslant x}\tau(f_k(n))\\ &\ll_k x\sum_{m\leqslant x}\frac{k(m)}{m}\\ &\ll_k x\prod_{q\leqslant x}\left(1+\sum_{i\geqslant 1}\frac{k(q^i)}{q^i}\right)\\ &\ll_k x\prod_{q\leqslant x}\left(1+\frac{k(q)}{q}\right)\\ &\ll_k x(\log x)^{\#I}.\end{aligned}$$

The above calculation is clear except for the inequalities between lines 2 and 3 which follows from Theorem 7.1 in [3] and the inequality between lines 5 and 6 which follows from the calculations performed at the lower bound. So, we only need to save a factor of one $\log x$ and this we will achieve using the fact that the sum we are interested in is only over primes $p\leqslant x$. To do this, we need an inequality for $\tau(f(p))$. Here it is:

Lemma 1. *Let $C>1$ be a fixed constant. For all but $O(x/(\log x)^C)$ primes $p\leqslant x$ we either have*

$$\tau(f_k(p)) \ll \sum_{\substack{m\leqslant x^{1/2}\\ m\mid f_k(p)}} 1, \tag{3}$$

or

$$\tau(f_k(p)) \leqslant O(1)^r \sum_{m\in S_r: m\mid f_k(p)} 1$$

for some $4\leqslant r\ll(\log\log x)^2$, where S_r is the set of all m with the following properties:

(i) *$m\in[x^{1/8},x^{1/2}]$;*
(ii) *m is $x^{1/r}$-smooth. That is, m is not a multiple of any prime $p>x^{1/r}$;*
(iii) *m has at most $(\log\log x)^2$ prime factors;*
(iv) *m is not divisible by any prime power p^k with $k\geqslant 1$, $p\leqslant x^{1/4}$ and $p^k\geqslant x^{\frac{1}{16(\log\log x)^2}}$.*

Proof. This is Lemma 7.3 in [3] and in fact it holds for any n not only just for primes. In that lemma, the range for m in (3) is $m\leqslant x$ (instead of $m\leqslant x^{1/2}$) while in (i) the range for m is $[x^{1/4},x]$ instead of $[x^{1/8},x^{1/2}]$, but one can see from the proof of Lemma 7.3 in [3] that the parameter N there can be replaced by $N^{1/2}$ which results in the current formulation, and the proof carries through. ∎

Armed with Lemma 1, we get that

$$\sum_{n\leqslant x}\tau(f_k(p))\leqslant\sum_{m\leqslant x^{1/2}}\sum_{\substack{p\leqslant x\\ m|f_k(p)}}1+\sum_{4\leqslant r\ll(\log\log x)^2}\sum_{\substack{m\in S_r\\ m|f_k(p)}}1+O\left(\sum_{n\in\mathcal{A}}\tau(f_k(n))\right),$$

where $\mathcal{A}$ can be taken to be a subset of $[1,x]$ of cardinality $x/(\log x)^C$ with an arbitrarily large C. By the argument from the proof of the lower bound, if C is large enough (say $C>d_k+d_k^2+4$), then the last term inside O can be made $O(x/(\log x)^2)$. For the first two terms in the right-hand side above, we fix $m\leqslant x^{1/2}$ and look at the congruence $f_k(p)\equiv 0 \pmod m$. This puts p into $k(m)$ residue classes modulo m, call them $a_1,\ldots,a_{k(m)}$. Thus,

$$\sum_{n\leqslant x}\tau(f_k(p))\leqslant\sum_{m\leqslant x^{1/2}}\sum_{i=1}^{k(m)}\pi(x,m,a_i)+\sum_{4\leqslant r\leqslant(\log\log x)^2}\sum_{m\in S_r}\sum_{i=1}^{k(m)}\pi(x,m,a_i)$$
$$+O(x/(\log x)^2).$$

For $\pi(x,m,a_i)$, we use the Brun–Titchmarsch inequality to deduce that

$$\pi(x,m,a_i)\ll\frac{x}{\phi(m)\log(x/m)}\ll\frac{x}{\phi(m)\log x}.$$

Hence,

$$\sum_{n\leqslant x}\tau(f_k(n))\ll\frac{x}{\log x}\left(\sum_{m\leqslant x^{1/2}}\frac{k(m)}{\phi(m)}+\sum_{4\leqslant r\leqslant(\log\log x)^2}\sum_{m\in S_r}\frac{1}{\phi(m)}\right)+O(x/(\log x)^2).$$

The first sum above gives

$$\sum_{m\leqslant x^{1/2}}\frac{k(m)}{\phi(m)}\leqslant\prod_{q\leqslant x^{1/2}}\left(1+\sum_{i\geqslant 1}\frac{k(q^i)}{\phi(q^i)}\right)\ll_k\prod_{q\leqslant x}\left(1+\frac{k(q)}{q-1}\right)$$
$$\ll\exp\left(\sum_{q\leqslant x}\frac{k(q)}{q}+O_k(1)\right)\ll(\log x)^{\#I}\ll(\log x)^k,$$

by estimate (2). As for the second sum, the argument on the second half of page 79 in [3] shows that

$$\sum_{m\in S_r}\frac{1}{\phi(m)}\ll_k\sum_{t=1}^{\infty}\frac{O(1)^{rt}}{\lfloor rt/100\rfloor!}\left(\sum_{x^{1/(2^{t+1}r)}\leqslant p\leqslant x^{1/(2^t r)}}\frac{1}{p-1}\right)^{\lfloor rt/100\rfloor}\left(\sum_{u\leqslant x}\frac{k(u)}{\phi(u)}\right),$$

therefore

$$\sum_{4\leqslant r\leqslant(\log\log x)^2}\sum_{m\in S_r}\frac{1}{\phi(m)}\ll_k\left(\sum_{r=2}^{\infty}\sum_{t=1}^{\infty}\frac{O(1)^{rt}}{\lfloor rt/100\rfloor!}\right)\sum_{u\leqslant x}\frac{k(u)}{\phi(u)}\ll\sum_{u\leqslant x}\frac{k(u)}{\phi(u)}$$

and the last sum is $O((\log x)^k)$ again by estimate (2). This finishes the proof of the upper bound. For the first sum, the argument is identical except that there $I = \{1, 2, \ldots, k\}$.

Acknowledgements. We thank the referee for suggestions which improved the quality of the paper. This paper started during a visit of F.L. to the Università Roma Tre, Italy in March, 2019 partially supported by the grant "Dipartimento di Eccellenza 2018-2022" and ended during a visit of F.L. to the Max Planck Institute for Mathematics in Bonn, Germany from September 2019 to February 2020. F.L. thanks these institutions for their hospitality and support. F.P. was supported in part by G.N.S.A.G.A. from I.N.D.A.M.

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Addresses: Florian Luca: School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa and Research Group in Algebraic Structures and Applications, King Abdulaziz University, Jeddah, Saudi Arabia and Centro de Ciencias Matemáticas UNAM, Morelia, Mexico;
Francesco Pappalardi: Dipartimento di Matematica e Fisica, Università Roma Tre, Largo S. L. Murialdo 1, I–00141 Roma, Italy.

E-mail: Florian.Luca@wits.ac.za, pappa@mat.uniroma3.it

Received: 4 February 2020; **revised:** 14 July 2020